Int. J. Heat Mass Transfer. Vol. 14, pp. 2051–2062. Pergamon Press 1971. Printed in Great Britain

LAMINAR AND AXISYMMETRIC VERTICAL JETS IN A STABLY STRATIFIED ENVIRONMENT

A. R. TENNER
Esso Research and Engineering, Florham Park, N.J., U.S.A.
and
B. GEBHART
Professor of Mechanical Engineering, Cornell University, Ithaca, N.Y., U.S.A.

(*Received* 22 *December* 1970 *and in revised form* 17 *March* 1971)

Abstract—Experiments have shown that an axisymmetric, laminar, buoyant jet in a stably stratified environment induces the flow of a toroidal cell around itself. The inner portion of the cell is driven upward by the viscous shearing of the jet, and the outer portion descends due to a negative buoyancy force. Under certain limiting conditions, this cell draws along a thin layer of the lower density jet and surrounds itself with it in the form of a shroud. The shroud flows in a direction opposite to the jet, and its density is similar to that of the jet. Conditions favorable to shroud production require that the environment be stably stratified and that the molecular diffusivities of the fluids involved be extremely small. Turbulent jets are not expected to produce an appreciable shroud.

INTRODUCTION

RECENT concern about air and water pollution has caused considerable interest in the flow of buoyant materials in stably stratified environments. The fluids in man's natural surroundings are stratified. Smoke from a stack flows into the air as a buoyant jet, as does sewage that is pumped into quiescent or running water. Since such flows, as well as those associated with natural phenomena, are often turbulent, little work has been done concerning laminar flows. Due to the tremendous efficiency of turbulent mixing, such a jet mixes quite well with its entraining environment. If pollutants are discharged in this way, the effectiveness of any attempt to rid the local surroundings of them is limited. One might consider the possible advantage of emitting them as laminar jets. This study was motivated by the need to learn more about the flow of a laminar jet into a stably stratified environment.

A theoretical model of the laminar, axisymmetric, vertical plume in a stably stratified environment was formulated by Morton [6]. He considered fluids with a Prandtl number equal to or greater than 1, and plumes which were generated solely by thermal sources; that is, plumes driven solely by a buoyancy force, without the introduction of mass or momentum at the source. The analysis was based on a postulated entrainment model, and accuracy was not expected to be any better than an order of magnitude.

The technique employed in Morton's analysis of the laminar plumes was the same as used by Morton *et al.* [7], and again by Morton [5], in investigations of turbulent forced plumes. In all of these studies the plumes were assumed to have profiles of vertical velocity and buoyancy which were similar at all heights. In some cases these profiles were assumed to decay exponentially from a maximum at the axis (Gaussian), and in others they were assumed uniform across the plume and zero outside ("top hat"). The entrainment rate scaled the mass inflow at the edge of the plume to the vertical velocity at that level, and was assumed to be proportional to some flow parameter.

Whereas the profiles and entrainment rates scale with the velocity in the turbulent plumes, they were found dependent on the Reynolds number in the laminar ones. In all three studies, the steady-state equations conserving mass, momentum, and buoyancy were non-dimensionalized and solved for the assumed profiles.

Loh-Nien Fan [3] experimented with turbulent jets; one was axisymmetric and vertical in a stagnant, linearly, and stably stratified salt water environment. This study of a turbulent jet is mentioned here because the stratification technique and non-dimensionalization schemes used are similar to those employed in the present study.

Fan achieved density stratification by filling the tank, layer by layer, with solutions of different densities. The heaviest solution was poured in first, and additional ones were gently introduced through a floating, plastic tube. Each new layer was lighter than the previous one and spread out horizontally on top. Each layer was an inch or two thick, but the resulting density profiles did not remain step-like because of molecular diffusion as well as mixing resulting from the disturbances introduced by the inflow of each layer. Local density was determined by a hydrometer.

Fan found that vertical turbulent jets could be characterized by two parameters, the Froude and Stratification numbers. The turbulent mixing process is an efficient and violent one, and the effects of turbulent diffusion are usually much larger than those resulting from molecular processes. Therefore, most physical quantities scale to velocity, or to turbulent intensity, regardless of the particular chemical, thermal, or momentum diffusivities involved. For a laminar flow, additional parameters will arise to completely specify the flow. These parameters relate the various molecular diffusivities which regulate laminar flow and diffusion, and they are determined from the set of governing equations.

This study will concern itself only with the steady-state flow of fresh and salt water jets in a stably stratified salt water environment. This allows the following assumptions: that density differences are related only to salt content, and that all other properties are constant. The flow is considered to be quasistatic, and the effect of changing density on continuity is assumed negligible.

The equations conserving mass, momentum, and chemical species are written for an axisymmetric and laminar flow field as follows:

$$\nabla \cdot \bar{w} = 0 \tag{1}$$

$$\bar{w} \cdot \nabla v = -(1/\rho)\, p_r + \nu \nabla^2 v - \nu \frac{v}{r^2} \tag{2}$$

$$\bar{w} \cdot \nabla u = -(1/\rho) p_x + \nu \nabla^2 u + g(\rho_\infty - \rho)/\rho \tag{3}$$

$$\bar{w} \cdot \nabla c = D \nabla^2 c \tag{4}$$

where c is the local salt concentration, and ρ_∞ is the density of the environment at the particular vertical location. One notes from the appearance of ρ_∞, that the effects of hydrostatic pressure variation have been included in the buoyancy term of equation (3), thus permitting the isolation of the pressure field associated with motion, p.

The diffusion of salt into water is relatively slow, and the density of a large portion of the jet remains constant at its initial value defined as ρ_j. Taking ρ_0 as the density of the stratified environment at the level of the nozzle, u_j as the average velocity of the jet at that level, and d as the nozzle diameter, it is convenient to non-dimensionalize all physical quantities as follows:

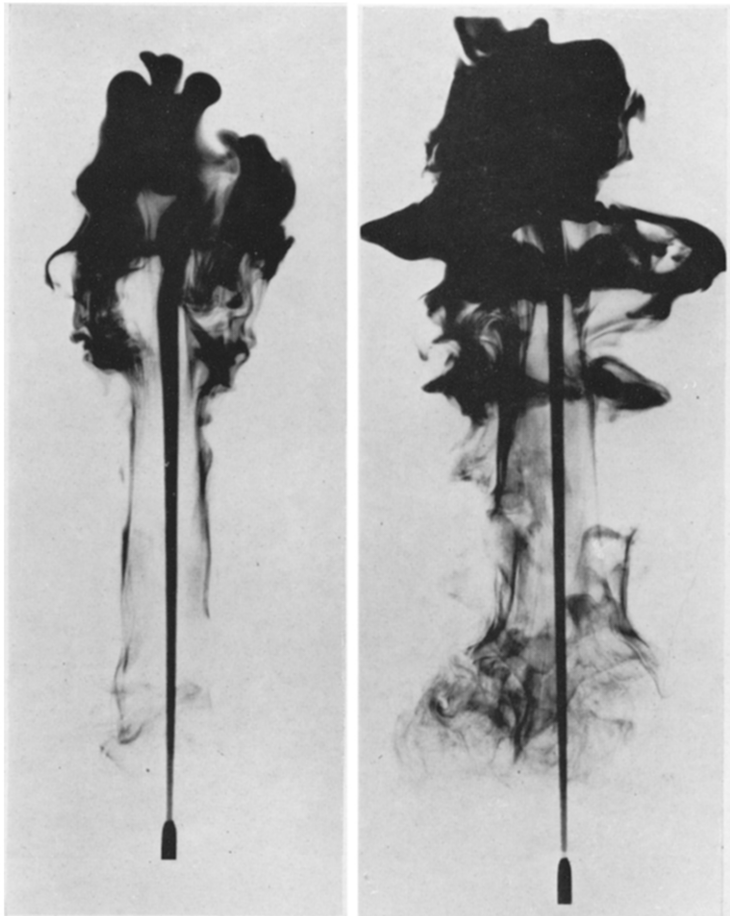
$$X = x/d \qquad R = r/d$$

$$U = u/u_j \qquad V = v/u_j \qquad \bar{W} = \bar{w}/u_j$$

$$P = \frac{p}{\rho u_j^2/2} \qquad M = \frac{\rho_0 - \rho}{\rho_0 - \rho_j}.$$

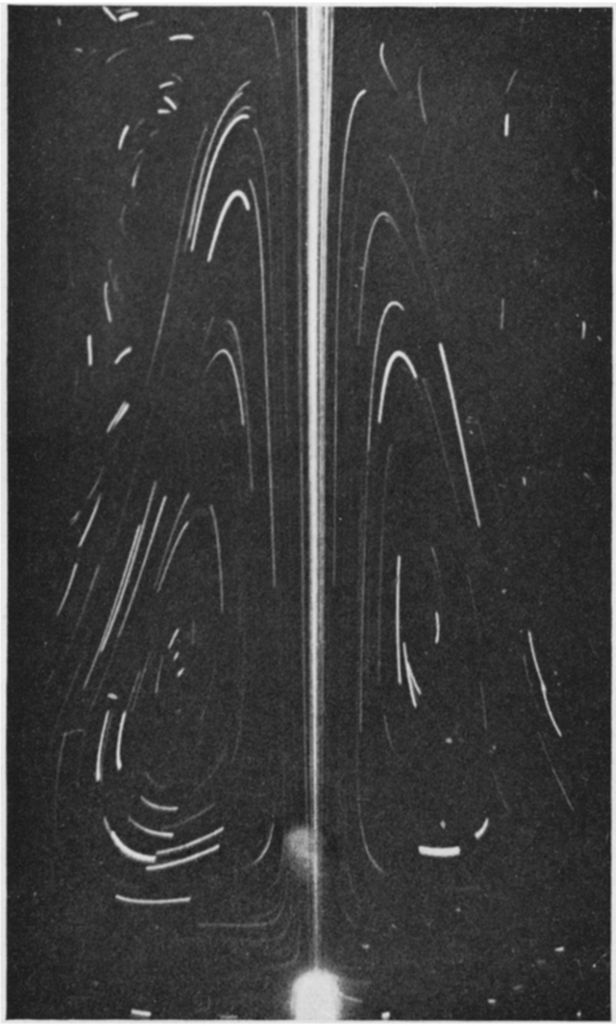
The equivalent to the Boussinesq approximation, used in thermally buoyant flows, is to assume that differences in density are small and linearly proportional to c, the salt concentration. Thus, c may be written simply in terms of ρ or M.

The above nondimensionalization scheme



Flow after 1/2 min   Flow after 1 min

FIG. 1. Jet and shroud as shown by dye.



15·83 s exposure

FIG. 2. Time exposure of particles suspended in flow.

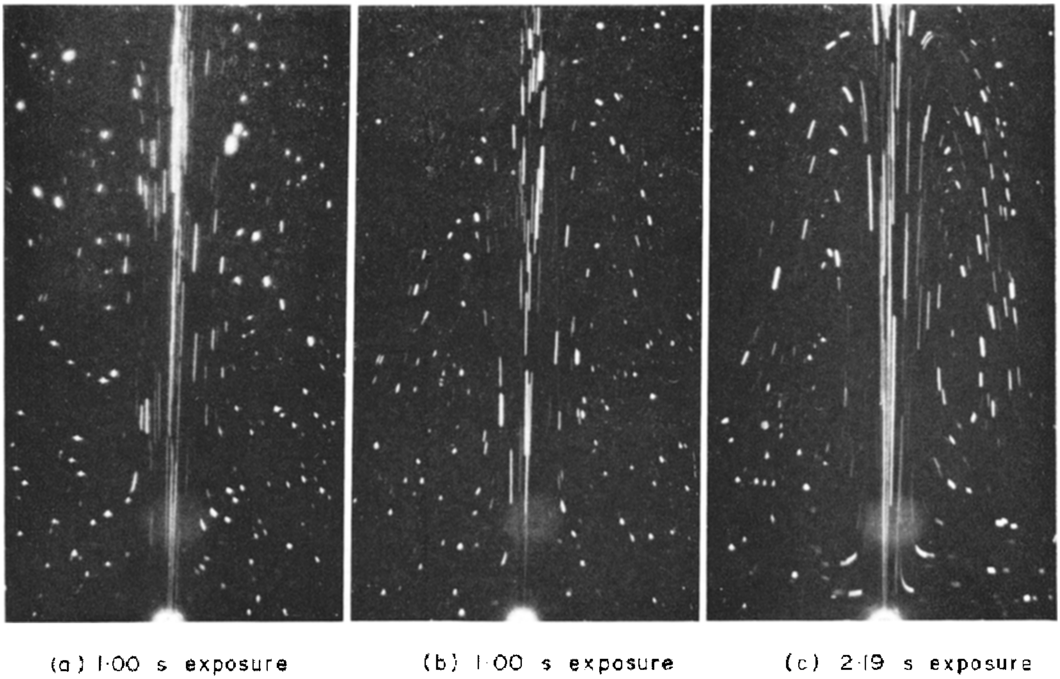


FIG. 3. Photographs used in determination of velocity profile.

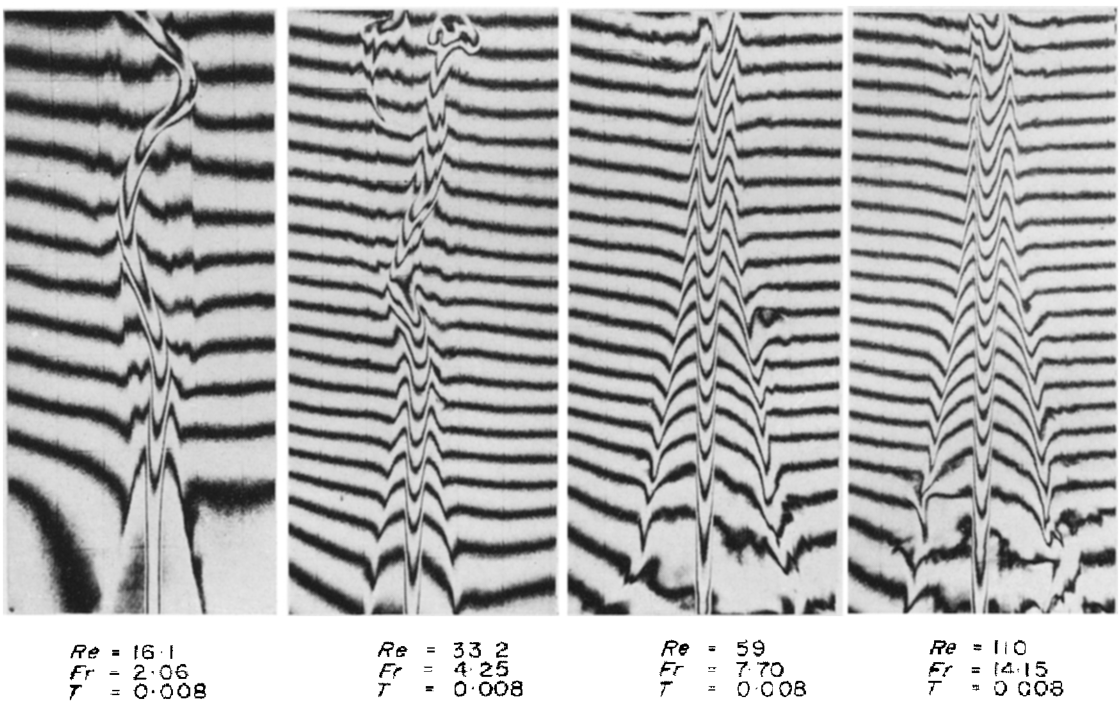
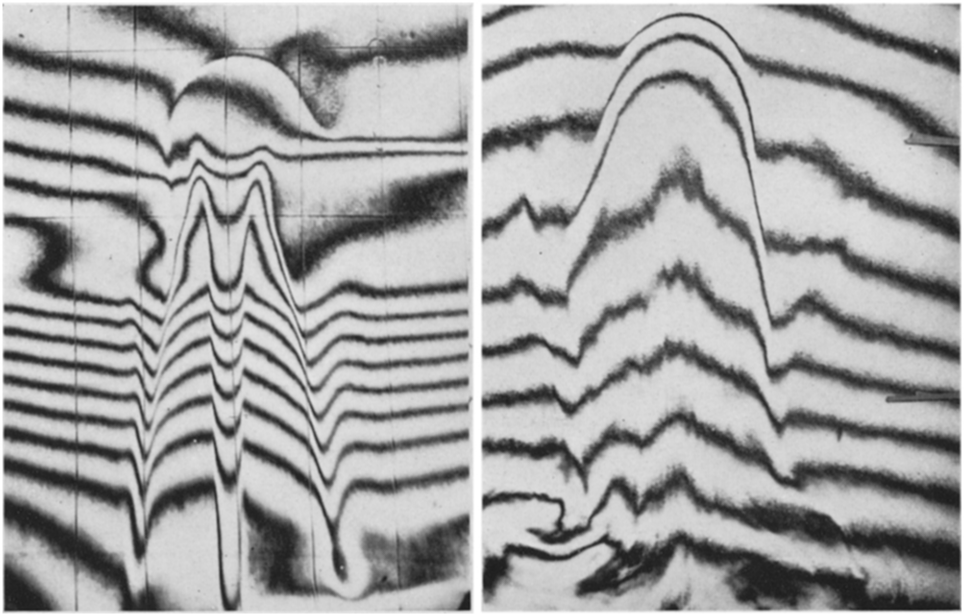


FIG. 5. Interferograms of four flow fields.



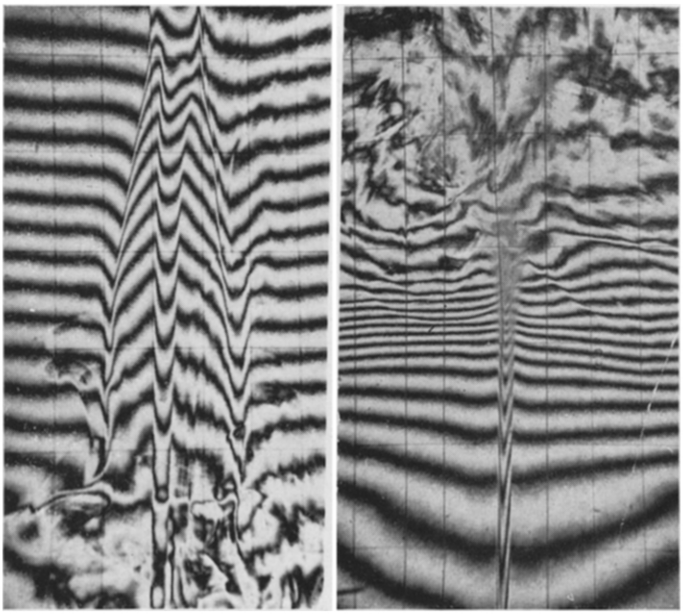
(a) $Re = 80$, $Fr = 3{\cdot}24$, $T = 0{\cdot}054$

Overshoot of jet with positive buoyancy

(b) $Re = 145$, $Fr = \infty$, $T = \infty$

Jet with zero initial buoyancy

FIG. 8. Interferometric observation of overshoot.



(a) $Re = 136$
$Fr = 30$
$T = 0{\cdot}012$

Shroud becomes unstable at Reynolds numbers above 100

(b) $Re = 572$
$Fr = 57$
$T = 8{\cdot}006$

Shroud is not observed in flows if jet is turbulent

FIG. 9. Interferometric observation of turbulence.

results in the following dimensionless parameters:

$$Re = u_j d/\nu$$

$$Sc = \nu/D$$

$$Fr = \frac{u_j}{\sqrt{[gd(\rho_0 - \rho_j)/\rho_j]}}$$

$$T = \frac{-(\rho_\infty)_X}{(\rho_0 - \rho_j)}.$$

The stratification number above, T, is equivalent to the inverse of the nondimensional stratification length. For linear stratification, this length is defined as the number of nozzle diameters above the nozzle where the density of the far field is the same as that of the lighter, jet material; that is, where $\rho_\infty = \rho_j$.

This is the level of neutral buoyancy. With the sign convention employed, T is positive for a relatively light jet flowing upward (or a heavy jet flowing downward) in an environment in which the density decreases with height (stable stratification).

For convenience, only flow fields in which the environment can be approximated as linearly stratified will be considered. By doing this, the distribution of stratified density is related to X linearly through T. If a co-ordinate system is defined such that $X = x = 0$ at the nozzle (where $\rho_\infty = \rho_0$), then the following simplification results:

$$\frac{\rho_\infty - \rho_0}{\rho_0 - \rho_j} = -TX.$$

The governing equations are written below for flow in a linearly stratified environment, in terms of the previously defined quantities.

$$\nabla \cdot \overline{W} = 0 \tag{5}$$

$$\overline{W} \cdot \nabla V = (1/Re)\nabla^2 V - (1/Re)\frac{V}{R^2} - \tfrac{1}{2}P_R \tag{6}$$

$$\overline{W} \cdot \nabla U = (1/Re)\nabla^2 U - \tfrac{1}{2}P_X + (\rho_j/\rho)(1/Fr^2)(M - TX) \tag{7}$$

$$\overline{W} \cdot \Delta M = (1/ReSc)\nabla^2 M. \tag{8}$$

The effects of buoyancy appear in the last term of equation (7). The quantity (ρ_j/ρ), appearing in this term, is very close to unity as long as density variations are small. In our experiments ρ_j/ρ is in the range from 0·997 to 1·0 and is assumed equal to one in subsequent use of the equations.

The flow involves four equations and four unknown distributions: U, V, M and P. The equations are strongly coupled, making analysis very difficult. Experimentation seems the best initial approach to understanding the flow. As shown by the above equations, the nature of the flow is completely specified by the parameters: Re, Fr, $ReSc$ and T.

EXPERIMENTAL FINDINGS

Three separate experimental methods were used in our study of such jets, each designed to determine a different characteristic of the flow field. In order to see initially how the jet flowed, and where the jet fluid eventually went, a set of experiments were conducted in which Nigrosin dye was used in the jet. A subsequent series of tests were run to determine the velocities of the jet and surrounding fluid. To accomplish this, Pliolite particles of about 0·005 in. dia. were suspended in the water of both the jet and the environment, and time exposures of the flow taken. Finally, an interferometric study was carried out to determine the density distribution in the flow field and in the surrounding region.

All experiments were performed in a tank internally twelve inches deep, six inches wide, and three inches thick. The jet nozzle opening was one-sixteenth of an inch in diameter, two inches from the bottom of the tank, and permanently aimed vertically upward. The environment was a stably stratified, salt water solution with never more than three parts of salt per thousand parts of water. The water depth was kept constant in the tank by an overflow. A wire grid was used in the field as a location reference scale.

Stable stratification was accomplished by a method similar to that used by Fan except that, in our study, successively heavier salt water solutions were introduced at the bottom of the tank. The local density, or salt concentration, in the undisturbed region of the tank was determined locally by electrical resistivity measurements of the solution. The stratification quickly became linear to the accuracy of our instruments and remained so.

The nozzle velocity of the jet was held constant by maintaining a constant difference in level between the free-surface of the jet reservoir and the free surface of the test tank. The velocity was controlled so as to produce a laminar flow. Reynolds number limits for laminar jets have been suggested by Da C'Andrade [2], Sato [9], Batchelor and Gill [1], Reynolds [8] and Viilu [10]. Depending on the criterion chosen, the predicted laminar limit ranges from $Re = 30$ to $Re = 300$. We were not concerned with the exact limit, and our experiments were limited to a range of Reynolds numbers which produced jets which appeared smooth and laminar. For the most part, the experiments were performed over the following range of the relevant parameters:

$$9{\cdot}1 \leqslant Re \leqslant 136$$

$$Sc = 635$$

$$0{\cdot}175 \leqslant Fr \leqslant 39{\cdot}3$$

$$0{\cdot}007 \leqslant T \leqslant 0{\cdot}054.$$

The main feature of the resulting flows was a tall, slender, laminar jet ending in a spreading cloud-like formation at approximately the level of neutral buoyancy of the initial jet material. In addition, experiments performed with dye showed the formation of a shroud flowing downward at some distance from the jet and ending in a region which we call the shroud base (see Fig. 1).

In the region between the jet and the shroud, the flow was that of a tall, narrow, toroidal cell. The streamlines in the plane of the jet for a typical cell are shown in Fig. 2. The result of viscous drag is that the flow adjacent to the jet is moving upward at a velocity comparable to that of the outer edge of the jet. By the same argument, the outer edge of the torroidal cell flows downward at the same velocity as the inner edge of the shroud beyond. Outside of this shroud, the field appears almost motionless. The actual velocity of the fluid may be ascertained from time exposures taken of the particles suspended in the flow. Figure 3 shows a set of photographs for one typical flow. Figure 4 is a

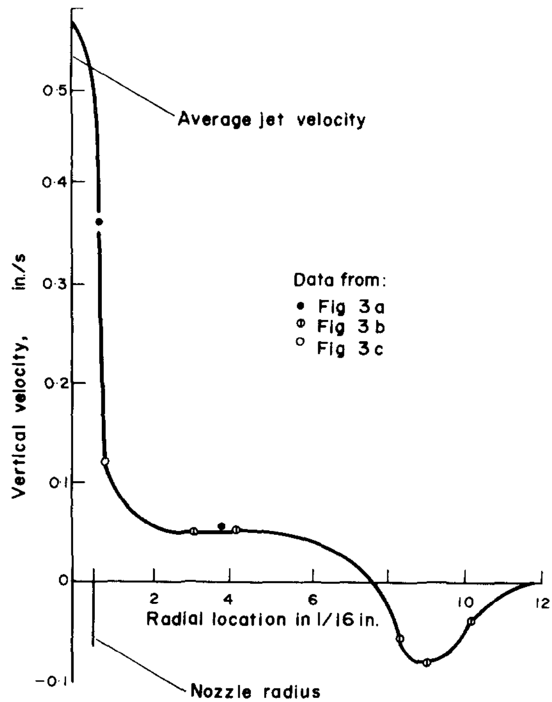


FIG. 4. Vertical velocity as a function of radial location at $1\frac{1}{2}$ in. above the nozzle for a typical flow, as calculated from particle traces.

velocity profile calculated from such photographs for a given jet. Some additional details concerning velocity determination are given in Appendix A.

The toroidal cell is drawn upward by the viscous action of the jet. This fluid becomes relatively heavier than the surrounding stratified fluid. Eventually, the negative buoyancy force exceeds the upward shear force, and the fluid reverses direction and flows down, to complete the cellular motion.

The small density differences in these flows. combined with the short effective optical path length through the axisymmetric flow field, made it impossible to determine the exact density variation through the flow region by interferometric fringe interpretation. However, based on the data available and on the knowledge gained from experiments conducted with dye and particles, a simple density field has been postulated as discussed in Appendix B.

A qualitative evaluation of the density field indicates that the shroud, which is flowing downward outside of the cell, is made up of relatively light fluid from the jet. This shroud must be thought of as a layer of lighter material being dragged down by the outer edge of the falling cell. The shroud appears to separate from the cell near the bottom, as the cell turns inward. After this separation, the light material of the shroud apparently turns outward and slowly mixes with the ambient field while attempting to rise due to the positive buoyancy force still exerted on it.

Calculations based on the postulated density field indicated that the density of the fluid in the toroidal cell is only slightly less than ρ_0, that is, the density of the stratified medium at the level of the nozzle. Given the magnitude of experimental errors, the density throughout the cell may be assumed equal to ρ_0 without contradicting interferometric data. Figure 5 shows four representative interferograms. Their evaluation is discussed in Appendix B.

RESULTS

This flow is a complex one and can perhaps be best thought of as seven separate ones coupled together at their adjacent boundaries. Figure 6 shows the seven postulated zones. This section will discuss each in detail.

Zone 1, *jet core*

In this zone there is no salt diffusion. Thus, the flow involves three unknowns (U, V and P), the three equations (5)–(7), and the boundary conditions imposed by the inner edge of Zone 2.

Consider flows in which the jet core extends a distance on the order of $1/T$ axially and 1 radially. If variations in velocity are smooth and regular, a relationship between axial and radial velocity components may be estimated from the continuity considerations,

$$\frac{V}{U} \sim \frac{R}{X}$$

which implies:

$$\frac{V}{U} \sim \frac{1}{1/T} = T.$$

This information can be used to assign an order of magnitude to the terms in the momentum equations, which are written out below, along with the associated orders of magnitude for the various terms. Recall that $O = 0(1)$.

$$VV_R + UV_X = \frac{1}{Re}\left(\frac{1}{R}V_R + V_{RR} + V_{XX} - \frac{V}{R^2}\right) - \tfrac{1}{2}P_R \qquad (9)$$

$$T^2 \quad T^2 \quad \frac{T}{Re} \quad \frac{T}{Re} \quad \frac{T^3}{Re} \quad \frac{T}{Re} \quad P$$

$$VU_R + UU_X = \frac{1}{Re}\left(\frac{1}{R}U_R + U_{RR} + U_{XX}\right) - \tfrac{1}{2}P_X + \frac{1}{Fr^2}(M - TX) \qquad (10)$$

$$T \quad T \quad \frac{1}{Re} \quad \frac{1}{Re} \quad \frac{T^2}{Re} \quad PT \quad \frac{1}{Fr^2}$$

The magnitude of the motion pressure may be ascertained from equation (9). In most cases of practical interest T is much less than one (the stratification length is much greater than the nozzle diameter), and the largest term of the radial momentum equation is either of the order T^2 or T/Re, depending on the relative magnitude of T and Re. If $Re \geqslant 1/T$, $O(P) = O(T^2)$, or if $Re < 1/T$, $O(P) = O(T/Re)$.

The magnitude of the terms in the axial momentum equation is the same as in the radial momentum one, multiplied by a factor of $1/T$. Therefore, if T is very much less than one, the radial effects are much less than the axial, and may be ignored. On the other hand, if $O(T) \geqslant 1$, radial momentum is important and may not be ignored. The physical significance of this is that if the stratification is strong the jet cannot rise very far before buoyancy becomes negative. Such a jet must begin to flare out quite fast and reverse its flow direction and radial momentum is obviously important. If the stratification is weak, $T \ll 1$, radial momentum is unimportant, and motion pressure in the jet core may be considered to vary with X only. Examination of the relative magnitude of the terms in equations (9) and (10) shows this to be true even for low Reynolds number flows. The remainder of this discussion will deal with such weak stratification, thus radial variations in pressure and momentum are negligible in the jet core.

We will now examine how the pressure term compares in size to the other terms in equation (10). Using the order of P determined from the analysis of equation (9), if $Re \geqslant 1/T$, $P_X = O(T^3)$, or if $Re < 1/T$, $P_X = O(T^2/Re)$. In either case, this suggests that the motion pressure term differs from the dominant terms of the axial momentum equation by an order of T^2. Experiments performed with particles suspended in the flow indicate that the velocities outside of the jet core are on the order of one tenth those in the core (see Fig. 4). The pressure variations due to motion outside of the jet core, i.e. in the cell, would be of negligible order in the jet and it may be assumed that the pressure in the jet is equal to the local hydrostatic pressure, for $T \ll 1$.

Axial momentum is, therefore, affected by only two forces: viscous shearing and buoyancy. For certain values of the Froude and Reynolds numbers, some terms become relatively unimportant, and the axial momentum equation may be written in a simplified form. The following table covers the range of possible combinations of the size of the parameters, and the characterizations of the different classes of jet flow resulting. The discussion is restricted to flows for which $T \ll 1$. The letters denote regimes of flow.

	$Re \ll O(1/T)$	$Re = O(1/T)$	$Re \gg O(1/T)$
$Fr^2 \ll O(1/T)$	a	b	b, c
$Fr^2 = O(1/T)$	d	e	c
$Fr^2 \gg O(1/T)$	d, f	f	c

Note that for the Froude number much smaller than one, the velocity induced by buoyancy may be greater than u_j, and the assumption that $U = O(1)$ may no longer be correct. In this event, the assigned orders of magnitude are incorrect and results based on them are invalid. Under certain conditions this is the case for classes (a) and (b).

Tall, thin laminar jets, in a weakly stratified environment, are limited to classes (a), (b) and (e). For jets of class (a), the axial momentum equation reduces to:

$$\nabla^2 U = -(Re/Fr^2)(M - TX).$$

That is, viscous and buoyancy forces balance. The applicable equation for jets of class (e) is:

$$\overline{W} \cdot \nabla U = (1/Re)\nabla^2 U + (1/Fr^2)(M - TX).$$

For jets of class (c) flowing in a weakly stratified environment ($T \ll 1$), the Reynolds number is large and the flow turbulent. By the definition of classes (d) and (f), the Froude number must be much bigger than the Reynolds number. For any given fluid and nozzle diameter, the only way to achieve such a condition is to have a very small density difference between the jet and its environment. If this density difference is very small, the stratification number can be small only if the variation of ρ_∞ with X is very small. From a practical point of view, classes (d) and (f) must be considered to be non-buoyant jets flowing in a uniform environment.

Summarizing for tall and thin jets, flow in zone 1 is governed by considerations of continuity and conservation of axial momentum. The flow in the jet core is regulated by Re, Fr and T. By the way in which zone 1 is defined, (the region where $\rho \equiv \rho_j$) the Schmidt number only serves to locate its boundary. If the Schmidt number is small, salt diffusion is so rapid that

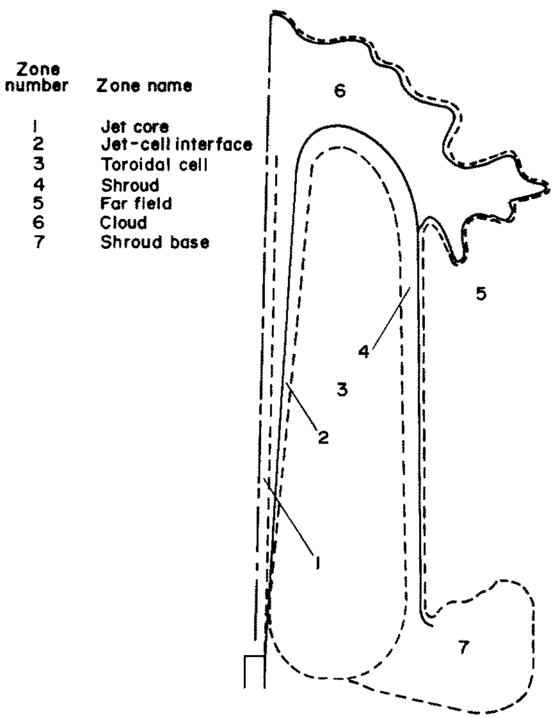


FIG. 6. Sketch of the seven flow zones.

this zone exists only near the nozzle. On the other hand, if the Schmidt number is very large, the density of the entire jet is uniform, and the core may be thought to extend all the way to the inner edge of the cell.

Zone 2, jet–cell interface

This is the thin region where molecular diffusion of salt occurs between the toroidal cell and jet. It is characterized by viscous forces and a large concentration gradient. This zone is increasingly thin as the Schmidt number increases. For $Sc = 635$, this region could probably be neglected and density taken as a step function from the jet density, ρ_j, to some other value at the inner edge of the cell.

Zone 3, toroidal cell

It has been postulated that the toroidal cell is formed by the jet dragging fluid up from the far field at the level of the nozzle. If this is true, the density of zone 3 must be essentially ρ_0, and the cell flow is described by three unknowns (U, V and P) governed by three conservation equations (mass, radial momentum, and axial momentum). As in zone 1, the flow in the toroidal cell depends on Re, Fr and T. The Schmidt number only serves to locate its boundaries.

In equilibrium, the upward shear stress must equal the negative buoyancy force. This balance may be used to estimate the overall size of this zone. Experiments indicate that a conical configuration is a reasonable approximation for the shape of the cell. Figure 7 shows a cross section of such a cone, and the forces acting on it.

Since all of the potential and kinetic energy of the jet is dissipated by viscous shearing, it seems likely that the total shear stress acting on the toroidal cell may be related directly to the jet flow. Since shear stress depends on viscosity and the local velocity gradient, we assume that this stress is proportional to the jet velocity. For flows in which the characteristic jet velocity is its nozzle velocity, that is, large Froude number, the total shear force may be related to u_j and to jet diameter d. Defining k as an unknown constant of proportionality relating the shear stress per unit length to the jet velocity, the total shear force per unit length is written as $\tau = \mu k u_j/d$. Using the boundaries for zone 3 that are shown in Fig. 7, assuming that the density throughout the zone is ρ_0, and using

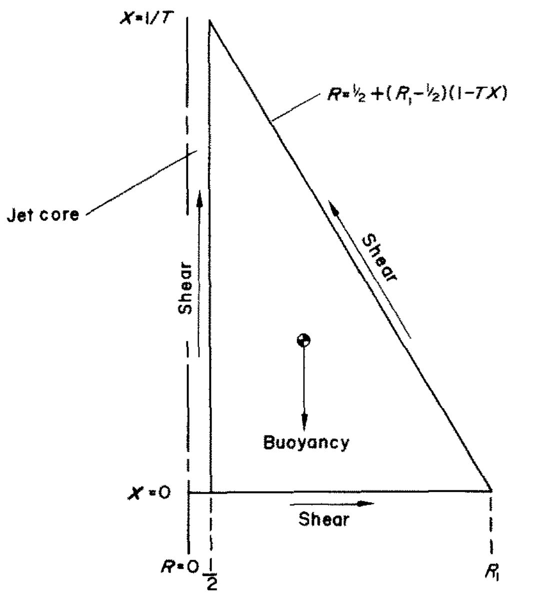


FIG. 7. Sketch of conical approximation to toroidal cell and forces acting on it.

a nondimensional length scale based on d, the force balance equating shear to buoyancy is written as follows:

$$\frac{\mu k u_j}{T} = \int_0^{1/T} \pi d \{[\tfrac{1}{2} + (R_1 - \tfrac{1}{2}) \times (1 - TX)]^2 - (\tfrac{1}{2})^2\} [(\rho_0 - \rho_\infty)g] \mathrm{d}X. \quad (11)$$

For linear stratification in the far field, equation (11) yields the following relationship between the size of the toroidal cell, R_1, and the parameters of the jet flow

$$R_1 = \sqrt{[1 + (12k/\pi)(Fr^2/Re)]} - \tfrac{1}{2}. \quad (12)$$

Although experimental data shows considerable scatter, jets in a weakly stratified environment generated flows in which the maximum radial extent of the cell could best be characterized by:

$$R_1 = \sqrt{[1 + 20(Fr^2/Re)]} - \tfrac{1}{2}.$$

This result implies that $k = (\tfrac{5}{3})\pi$. Other approximations for the shear stress may be used by modifying the assumed flow conditions. For example, the jet core diameter may be considered to vary with height.

Thus, the radial extent of the toroidal cell apparently depends on the ratio of Fr^2/Re. This quantity is the ratio of viscous force to buoyancy. For a given fluid and density difference $(\rho_0 - \rho_j)$, Fr^2/Re varies directly with the initial velocity of the jet. The increase of the radial extent of the toroidal cell with jet velocity (Fr^2/Re) may be seen in the interferograms of Fig. 5.

Zone 4, *shroud*

The flow in this region is similar to that of zone 2 in that it is characterized by considerable viscous shearing and salt diffusion. One main difference in the shroud, however, is that the buoyancy force opposes the direction of flow. In the shroud, radial momentum may be ignored (as long as the conditions which were discussed in regard to zone 1 are met), and, since shroud velocities are very small, pressure may be considered to be hydrostatic only. In this zone, three unknown distributions (U, V and M) are governed by three equations (5), (7) and (8).

Zone 5, *far field*

Observations made with particles indicate that immediately outside of the shroud a small amount of fluid is slowly dragged down. This fluid then circulates in the same type of cellular motion which characterizes the toroidal cell, zone 3. However, this effect in the far field is very small, i.e. the velocities and cell sizes are small. If this motion is ignored, the far field is simply a region of no flow and provides simple boundary conditions at the outer edge of the shroud.

Zone 6, *the cloud*

This region differs from the preceding five in that variations with time may not be disregarded. When the jet is started, the cloud does not exist; it grows with time. Flow in the cloud also includes the region of the momentum overshoot of the jet. This fluid then tends to flatten into a sheet at the level of neutral buoyancy. Since there is a finite amount of molecular diffusion, a finite size to the test tank, and since the density differences throughout the region are relatively small, a definite sheet is not formed. The early growth of the cloud with time is seen in Fig. 1.

Figure 8a shows the entire field, including the overshoot and spreading of the cloud. The dome shaped fringes at the top of the interferogram show the region of overshoot. Slightly below this, the fringes of the far field are spread out. This corresponds to the level of neutral buoyancy of the jet and reveals the spreading out of the netural density cloud here.

Zone 7, shroud base

Like the cloud, this region does not exhibit a steady-state behavior while the jet is maintained. Just as the cloud received most of the fluid from the jet, the shroud base receives the fluid from the shroud. Since this fluid is generally lighter than the surrounding far field, it tends to rise. However, due to diffusion and the convection associated with the upward motion, it slowly mixes into the surroundings. If a jet is run sufficiently long, the accumulation of fluid in zone 7 effectively changes what was considered to be the distant undisturbed region.

CONCLUSIONS

The flow of buoyant jets of the type considered here is completely defined by four parameters, *Re*, *T*, *ReSc* and *Fr*, and the shroud is observed only for certain limiting values of these parameters. This section will discuss the ranges over which each parameter may vary and maintain flows favorable to shroud production. If one desires to minimize the mixing of the jet with its environment, the production of a shroud must be avoided.

Reynolds number

The shroud is formed by the induced motion of the toroidal cell, which in turn is driven by the viscous shearing of the jet. For jets with Reynolds numbers below about 10, in stably stratified salt water, this shroud is practically undiscernible because its density is almost the same as that of the far field. This shroud is weak (has a small density deficiency) because it is diffused away almost as fast as it is formed. The relative importance of salt diffusion is related to the *ReSc* product.

Jets with Reynolds numbers over 100 are near the laminar limit, and the shroud tends to become unsteady, and to separate from the driving cell. Figure 9a shows the deterioration of the shroud at $Re = 136$. No shrouds were produced at all for Reynolds number greater than about 200 because of the onset of turbulence in the jet. The exact laminar limit for the jet is not expected to be a single value in such flows, but will likely depend on the Froude and Stratification numbers because of their effects on velocity, through buoyancy. Figure 9b shows a jet which has become completely turbulent at the top. Turbulent mixing has suppressed the formation of the shroud.

Stratification number

The cellular motion found in the region between the jet and the shroud will occur only if the density of the environment decreases with height (stable stratification). Since the formation of the shroud depends on the downward motion of the outer portion of the toroidal cell, stable stratification is a requirement for shroud production.

ReSc *product*

Unfortunately, the experiment did not permit a variation of the Schmidt number. However, this is not a serious drawback because the relative importance of salt diffusion is shown by the *ReSc* product, not by the Schmidt number alone. The only thing that may be said about Schmidt number dependence is based on experiments in the interferometer using thermal stratification and a warm water jet, rather than one of salt. (The parameters of this flow are the same as for the salt water one except the Prandtl number replaces the Schmidt number.) The shroud was not observed. The probable explanation of its absence is that there are about two orders of magnitude difference between thermal and salt molecular diffusion rates in water. At 70°F, for low salt concentrations, $Sc = 635$, while $Pr = 6{\cdot}81$. The ratio of these, the Lewis number, is 93. Thus, thermal differences diffuse 93 times faster than differences in salt concentration and, in a thermally buoyant flow, the shroud would be diffused away in the early stage of formation.

With respect to the importance of the molecular diffusion of salt, a jet with $Re = 100$ and $Sc = 63$, for example, is equivalent to one with $Re = 10$, $Sc = 630$. Our experiments showed that if the $ReSc$ product was small, the shroud was very weak; if this product was below about 6000, no shroud was detectable. Therefore, if the Schmidt (or Prandtl) number is 6, the Reynolds number must be about 1000 to produce a shroud. This would result in turbulence. However, the shroud is suppressed by turbulent mixing.

Assuming that a shroud is not produced by jets with an $ReSc$ product of less than about 6000 and setting the laminar limit as $Re < 200$, a minimum value of the Schmidt (or Prandtl) number for shroud production is 30.

Froude number

If the initial buoyancy of the jet is zero or negative, the flow is completely different than discussed above. Instead of producing the tall, narrow jet, the flow flares out and reverses its direction like a fountain as soon as the negative buoyancy becomes sufficient to overcome the initial momentum. The downward flowing jet looks like a shroud itself, and ends in a cloud at the level of neutral buoyancy. However, another shroud is formed outside of this one and flows in the opposite direction. This shroud is weak, irregular, and barely discernable, but is of the same nature as the one produced by the tall, narrow jet. Figure 8b shows these features for a flow in which the level of neutral buoyancy of the jet is at the nozzle and $Fr = \infty$. Careful examination of the area just outside of the bullet shaped region of the jet's flow reveals a fringe shift similar in form, but opposite to, that of the shroud in the interferograms in which Fr was not infinite.

For jets with positive buoyancy, the Froude number has effects similar to those of the Reynolds number. Jets which are very buoyant have relatively large convective velocities, and these add to the effects of the forced velocity associated with the Reynolds number.

Although the Froude number is extremely important in determining the type of flow, it in no way inhibits the formation of the shroud. Over the entire range of Froude numbers observed, shrouds could be produced as long as the other parameters were within the ranges suggested above.

Summary

There are apparently three restrictions imposed on the jet and environment in order for the jet flow to form a shroud.

(1) Flow must be laminar.
(2) Environment must be stably stratified.
(3) Chemical (or thermal) diffusion must be relatively slow.

ACKNOWLEDGEMENT

The authors wish to acknowledge support for this research under National Science Foundation Grants GK–1963 and GK–18529.

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APPENDIX A

Technique of Velocity Determination

Visualization of the flow field, and the measurement of velocities were accomplished by suspending Pliolite particles of about 0·005 in. dia. throughout the tank and jet water. A thin, vertical plane was illuminated in the plane of the jet. The beam of light was about $\frac{1}{8}$ in. thick by almost 3 in. wide.

Velocities were determined from time exposures. For exposures of one second or less, the accuracy of time measurement was the same as that of the built-in shutter speed of the camera. For longer exposures, the camera was set on bulb and a micro-switch was set to trigger when the shutter opened and again when it closed. The time elapsed was measured with a four digit timer.

Optimum results were obtained in measuring the velocity in and near the jet by exposures of less than one second, and velocities in the area of the shroud by exposures of several seconds. This was done on different negatives. The velocity profile of Fig. 4 is the combined result of data from three pictures taken in rapid succession. The flow is the same for all exposures. This was ensured by maintaining the head of the jet constant.

APPENDIX B

Technique of Density Determination

A 5 in. Mach–Zehnder interferometer was used to determine the density throughout the flow field. Since the flow was axisymmetric, the density variation does not relate to the fringe shift by a constant. The interferometer shows average density along the path of each light ray.

The local density may be determined by integration along the light path assuming a radial density variation. This is then iterated to achieve a variation which satisfies the interferogram. This technique, in numerical form, was used by Hossain [4]. However, in this study the density field is solved by a set of simultaneous equations, a simpler but less precise method. Use of this method is justified because the complexity of the axisymmetric, stratified flow field, the small density differences, and the short optical path length render a more exact interpretation of the data impractical.

Density profiles were assumed for each zone of flow. The jet was assumed to have a constant, known density ρ_j, the cell was assumed to have a constant density of unknown value, and the shroud was assumed to have a parabolic density variation. The far field at a given vertical location had a constant density determined by its salt content.

JETS VERTICAUX LAMINAIRES AXISYMÉTRIQUES DANS UN ENVIRONNEMENT STRATIFIÉ STABLE

Résumé—Des expériences ont montré qu'un jet axisymétrique laminaire avec effet d'Archimède dans un environnement stratifié stable induit le mouvement d'une cellule toroïdale autour de lui. La portion intérieure de la cellule est poussée vers le haut par la frottement visqueux du jet et la portion extérieure descend sous l'effet d'une force d'Archimède. Dans certaines conditions limites cette cellule entraîne une couche mince de plus faible densité que le jet et est entourée elle-même par une sorte de nappe. La nappe évolue dans une direction opposée à celle du jet et sa densité est semblable à celle du jet. Les conditions favorables à la production de la nappe exigent que l'environnement soit stratifié et stable, et que les diffusivités moléculaires des fluides soient extrêmement faibles. Les jets turbulents ne semblent pas devoir produire un effet appréciable.

EIN LAMINARER UND AXIALSYMMETRISCHER STRAHL IN EINER STABIL GESCHICHTETEN UMGEBUNG

Zusammenfassung—Experimente haben gezeigt, dass ein axialsymmetrischer, laminarer, aufwärtsstrebender Strahl in einer stabil geschichteten Umgebung die Strömung in einer ringförmigen Zelle um sich selbst induziert. Das innere Gebiet der Zelle wird durch die viskose Scherung des Strahls aufwärtsgetrieben. und das äussere Gebiet sinkt infolge einer negativen Auftriebskraft ab. Unter gewissen Randbedingungen zieht die Zelle eine dünne Schicht des Strahles mit niedriger Dichte an und umhüllt sich damit. Die Hülle strömt in einer Richtung entgegengesetzt zum Strahl, und seine Dichte ist entsprechend der des Strahls. Günstige Voraussetzungen für die Erzeugung der Hülle verlangen, dass die Umgebung stabil geschichtet ist und dass das molekulare Diffusionsvermögen der Fluide, das damit verwickelt ist, äusserst klein ist. Turbulente Strahlen lassen die Erzeugung einer merklichen Hülle nicht erwarten.

ЛАМИНАРНЫЕ И ОСЕСИММЕТРИЧНЫЕ ВЕРТИКАЛЬНЫЕ СТРУИ В УСТОИЧИВО СТРАТИФИЦИРОВАННОМ СРЕДЕ

Аннотация—Эксперименты показывают, что осесимметричная ламинарная струя, подверженная действию подъёмной силы, в устойчиво стратифицированной среде ввзывает кольцевое ячеистое течение вокруг себя. Внутренняя часть ячейки вытягивается вверх за счёт вязкого натяжения струи, а внешняя—опускается за счёт отрицательной подъёмной силы. При определённых ограничивающих условиях, эта ячейка перемешается вдоль тонкого слоя струи с малой плотностью и в свою очередь окружается чем-то в роде защитною экрана. Этот экран движется в направлении, противоположном струе, и его плотность аналогична плотности струи. Условия, благоприятные для создания защитной оболочки, требуют, чтобы окружающая среда была устойчиво стратифицированна и чтобы молекулярная диффузия рассматриваемых жидкостей были чрезвычайно малы. Ожидается, что турбулентные струи не создают значительной оболочки.